2.15 f) $\lim_{y \to 1} \frac{2^{-y} - \frac{1}{2}}{2^y - 2} = \lim_{y \to 1} \frac{\frac{1}{2^y} - \frac{1}{2}}{2^y - 2}$

$= \lim_{y \to 1} \frac{\frac{2}{2^y \cdot 2} - \frac{2^y}{2^y \cdot 2}}{2^y - 2} = \lim_{y \to 1} \frac{\frac{(2 - 2^y)}{2^{y+1}}}{(2^y - 2)/1}$

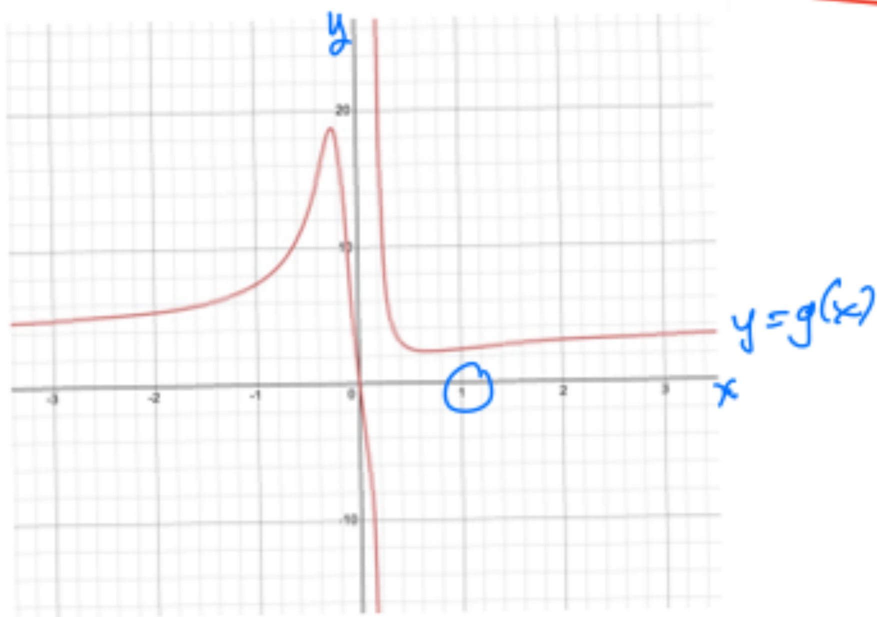
$= \lim_{y \to 1} \frac{(2 - 2^y)}{2^{y+1}} \cdot \frac{1}{(2^y - 2)} = \lim_{y \to 1} \frac{(-1)\cancel{(2^y - 2)}}{2^{y+1}\cancel{(2^y - 2)}} = \boxed{\frac{-1}{4}}$

2.15 j) $\lim_{s \to 1} \frac{|s^2 - 1|}{s^2 - 1} = \lim_{s \to 1} \frac{|(s-1)(s+1)|}{(s-1)(s+1)}$ = $\lim_{s \to 1} \frac{|s-1|\cancel{|s+1|}}{(s-1)\cancel{(s+1)}}$

↑ always +

$\lim_{s \to 1^-} \frac{|s-1|}{s-1} = \lim_{s \to 1^-} \frac{-(s-1)}{(s-1)} = -1$, $\lim_{s \to 1^+} \frac{|s-1|}{s-1} = 1$. Therefore, the limit DNE.

2.20



c) On what set is g continuous? $\mathbb{R} \setminus \{.2\} = (-\infty, .2) \cup (.2, \infty)$

↑ set of real #s

$= \{x \in \mathbb{R} : x < .2 \text{ or } x > .2\}$

Examples:

① If y is a constant, compute

$$L = \lim_{a\to 0} \frac{a}{\frac{1}{y} - \frac{1}{y-2a}}$$

$$L = \lim_{a\to 0} \frac{a}{\frac{(y-2a)-y}{y(y-2a)}} = \lim_{a\to 0} \frac{(a/1)}{\frac{-2a}{y(y-2a)}}$$

$$= \lim_{a\to 0} \frac{\cancel{a}}{1} \cdot \frac{y(y-2a)}{-2\cancel{a}} = \lim_{a\to 0} \frac{y(y-2a)}{-2}$$

$$= \frac{y^2}{-2} = \boxed{-\frac{y^2}{2}}.$$

② Find

$$L = \lim_{x\to 1} \frac{\sin(\pi x)}{\sqrt{x}-1}$$

$\frac{0}{0}$ form

Side calculation: $\sin(\pi(x-1)) = \sin(\pi x - \pi)$

$= -\sin(\pi x)$.

$\sin(\theta - \pi) = -\sin(\theta)$

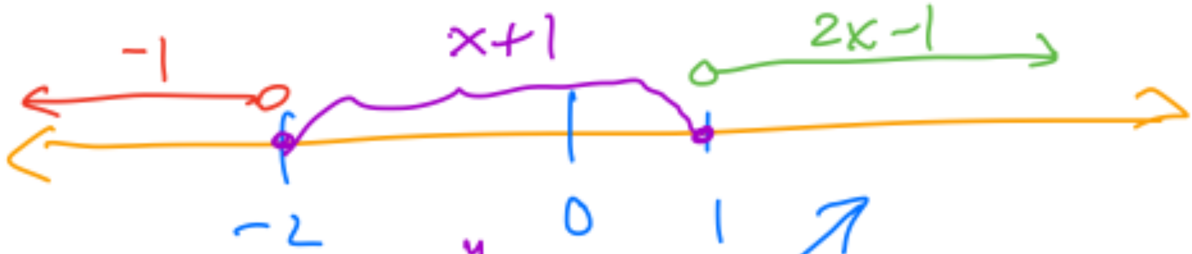
$$L = \lim_{x\to 1} -\frac{\sin(\pi(x-1))}{\sqrt{x}-1} = \lim_{x\to 1} -\frac{\overbrace{\frac{\sin(\pi(x-1))}{\pi(x-1)}}^{1} \cdot \pi(x-1)}{\sqrt{x}-1}$$

$$= \lim_{x\to 1} -\frac{\pi\overbrace{(x-1)}^{A^2-B^2}}{\sqrt{x}-1} = \lim_{x\to 1} -\frac{\pi(\cancel{\sqrt{x}-1})(\sqrt{x}+1)}{\cancel{\sqrt{x}-1}}$$

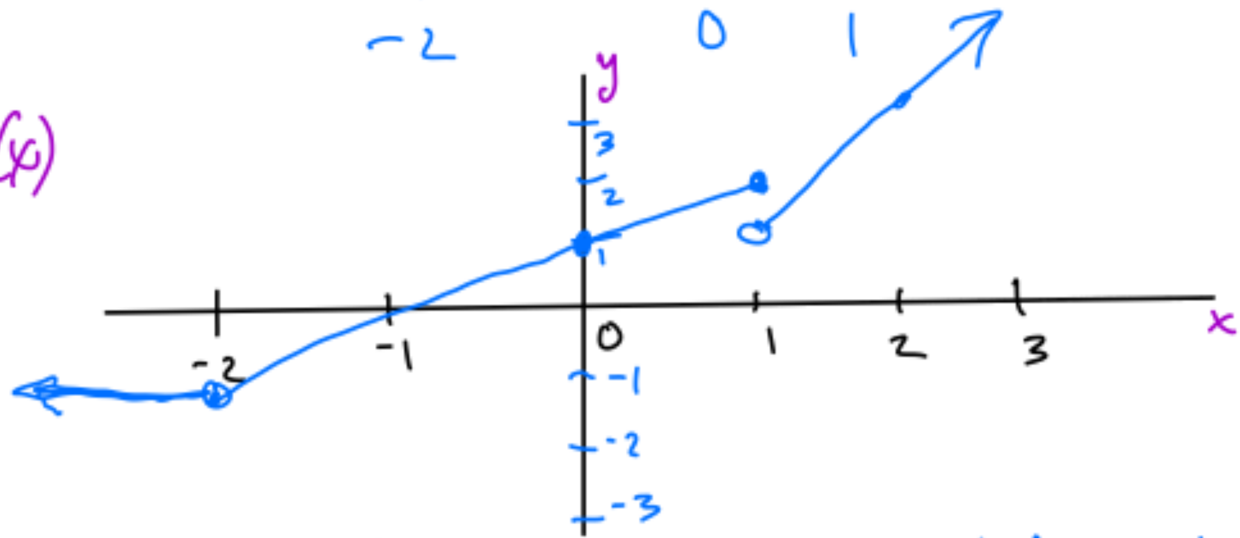
$$= \lim_{x\to 1} -\pi(\sqrt{x}+1) = \boxed{-2\pi}$$

$(A-B)(A+B) = A^2 - B^2$

(3) For which x is $\phi(x) = \begin{cases} x+1 & \text{for } -2 \le x \le 1 \\ -1 & \text{for } x < -2 \\ 2x-1 & \text{for } x > 1 \end{cases}$ continuous?



$y = \phi(x)$

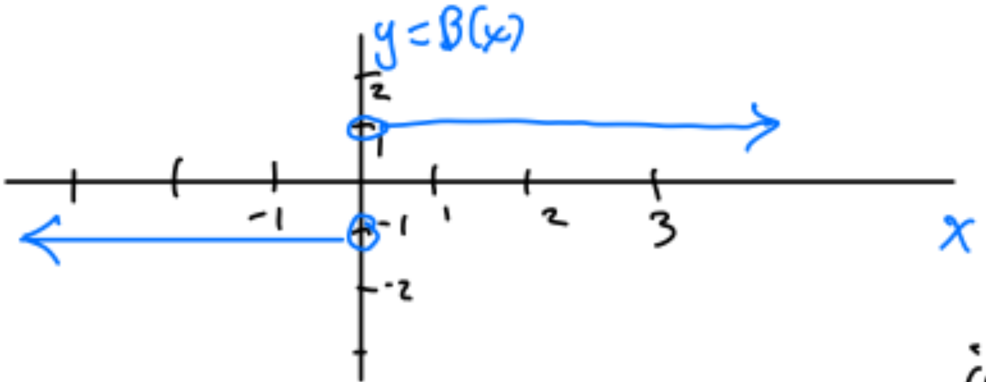


The only pt. where x is not continuous is at $x=1$. The set of discontinuity is $\{1\}$.
The set of continuity is $\mathbb{R} \setminus \{1\} = (-\infty, 1) \cup (1, \infty)$

Definition The function f is continuous at $x = a$ if a is in its domain, and

$$f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

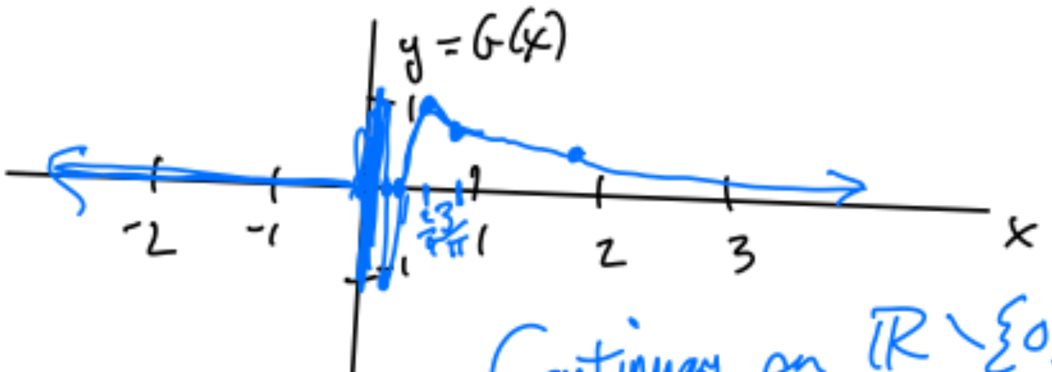
Example ④ Where is $B(x) = \frac{|x|}{x}$ continuous?
What does it look like?



Answer:
if $x \in \mathbb{R} \setminus \{0\}$,
$B(x)$ is continuous
at x.
i.e. B is continuous
on its domain
$= \mathbb{R} \setminus \{0\}$.

⑤ Let $G(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \end{cases}$.

Where is G continuous? What does it look like?



Continuous on $\mathbb{R} \setminus \{0\}$.

At 0: $\lim_{x \to 0^+} G(x)$ DNE (oscillates between -1 & 1)

$\lim_{x \to 0^-} G(x) = 0 = G(0)$

New calculus concept.

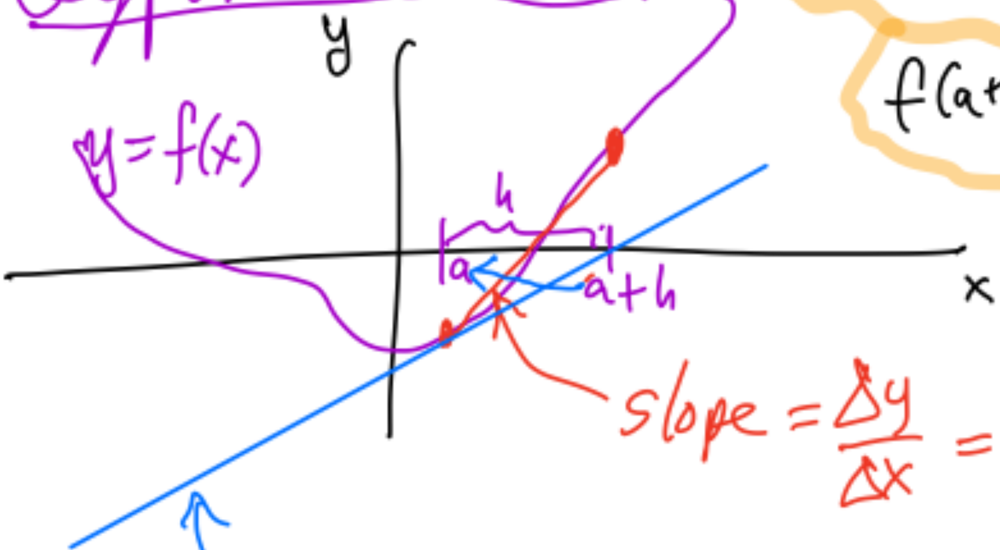
The **derivative** of a function f at a is

$$\frac{df}{dx}(a) = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If the limit exists, we say f is **differentiable at a**.

Think $h \simeq \Delta x$

$f(a+h) - f(a) \simeq \Delta y$



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

slope of tangent line $= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

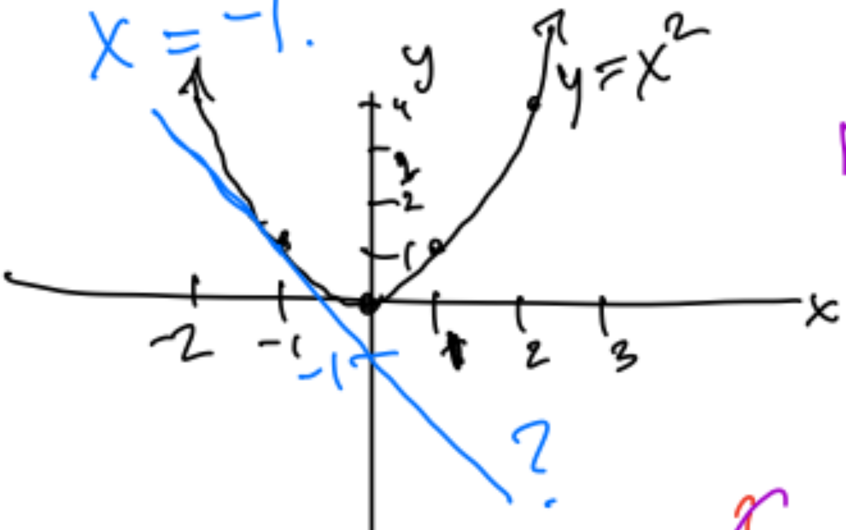
Interpretation:

$f'(a)$ = instantaneous rate of change of f at $x=a$ as x increases.

= slope of the tangent line to $y=f(x)$ at $x=a$.

Examples of using the definition to calculate tangent line slopes & equations.

(1) Using the definition, find the equation of the tangent line to $y=x^2$ at $x=-1$.



$f(x)=x^2$

$$m=\text{slope}=f'(-1)$$

$$=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

$$=\lim_{h\to 0}\frac{(-1+h)^2-(-1)^2}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{(-1)^2}+2(-1)(h)+h^2-\cancel{1}}{h}$$

$$=\lim_{h\to 0}\frac{-2h+h^2}{h}=\lim_{h\to 0}\frac{\cancel{h}(-2+h)}{\cancel{h}}=\boxed{-2}$$

$y-y_0=m(x-x_0)$ $\qquad (x_0,y_0)=(-1,1)$

$\boxed{y-1=-2(x+1)}$ $\qquad \boxed{y=-2x-1}$

$y-1=-2x-2$

2. Using a limit of slopes, calculate the slope of the tangent line to

$$y = \frac{x-3}{2x} \text{ at } (5, .2).$$

$$y' = \frac{df}{dx}(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(5+h)-3}{2(5+h)} - \frac{5-3}{2\cdot 5}}{h} \quad .2$$

$$= \lim_{h\to 0} \frac{\frac{2+h}{2(5+h)} - .2}{h} = \lim_{h\to 0} \frac{\frac{2+h - .2(2(5+h))}{2(5+h)}}{h} \quad \begin{matrix}(2+.4h) \\ .4(5+h)\end{matrix}$$

$$= \lim_{h\to 0} \frac{\frac{\cancel{2}+h-\cancel{2}-.4h}{2(5+h)}}{h} = \lim_{h\to 0} \frac{\frac{.6h}{2(5+h)}}{h/1}$$

$$= \lim_{h\to 0} \frac{.6\cancel{h}}{2(5+h)} \cdot \frac{1}{\cancel{h}} = \frac{.6}{10} = \boxed{.06}.$$

---

3. Calculate $g'(2)$ if $g(x) = \frac{1}{\sqrt{2x+1}}$

$$g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{2(2+h)+1}} - \frac{1}{\sqrt{5}}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sqrt{5} - \sqrt{5+2h}}{\sqrt{5}\sqrt{5+2h}}}{\frac{h}{1}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{5} - \sqrt{5+2h})}{\sqrt{5}\sqrt{5+2h}} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{(\overset{A}{\sqrt{5}} - \overset{B}{\sqrt{5+2h}})(\overset{A}{\sqrt{5}} + \overset{B}{\sqrt{5+2h}})}{\sqrt{5}\sqrt{5+2h} \cdot h\,(\sqrt{5} + \sqrt{5+2h})}$$

Multiply by the conjugate

$$(A-B)\frac{(A+B)}{A+B} = A-B$$

$$= \lim_{h \to 0} \frac{\overset{-2\cancel{h}}{\cancel{5-(5+2h)}}}{\sqrt{5}\sqrt{5+2h} \cdot \cancel{h} \cdot (\sqrt{5}+\sqrt{5+2h})}$$

$$= \frac{-2}{\underbrace{\sqrt{5}\sqrt{5}\,(\sqrt{5}+\sqrt{5})}_{5 \cdot 2\sqrt{5}}} = \frac{-2}{10\sqrt{5}} = \boxed{\frac{-1}{5\sqrt{5}}}$$

Quiz

1. What is your name?
2. What is your favorite movie?
3. Simplify $$\frac{\frac{1}{x+a} - \frac{1}{x}}{3a^5}$$
4. $$\lim_{\theta \to 0} \frac{\sin(2\theta)}{\sin(5\theta)}$$
5. Simplify $$\frac{A^2 - B^2}{3A^2 + 3AB}.$$